ON MINIMAL AND ALMOST-MINIMAL SYSTEMS OF NOTATIONS

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1. **Terminology.** We shall be concerned with well-orderings and well-founded partial orderings of the integers. When we come to deal with partial orderings $<_R$ which are not simple orderings (i.e., there are incomparable elements) in the later sections of this paper, we shall impose certain restrictions — e.g., there is to be a unique minimal element a_0 such that $a_0 <_R b$ for all b in the field of $<_R$ (henceforth denoted $F(R)$). We shall require that $a_0 = 1$. If $<_R$ is any partial ordering of the type being studied, then we define $| a |_R$ (read: "the ordinal of a in R", or simply "ordinal a", when it is clear what system $<_R$ is under consideration) by the following inductive definition:

(i) $| 1 |_R = 0$,

(ii) $| a |_R = \text{l.u.b.} \{ | b |_R : b <_R a \}$.

We shall also write "$| <_R |$" for "l.u.b. $\{ | a |_R : a \in F(R) \}$".

Our motivation is as follows: the systems $<_R$ under consideration are thought of as (many-one) *systems of notations* for ordinals in the second number class (cf. [K1], [W], [KR]). Each system $<_R$ contains notations for ordinals less than a certain countable ordinal, depending on the system—in fact, $<_R$ contains notations for just the ordinals less than $| <_R |$. If $a \in F(R)$, then (in the system $<_R$) we think of a as a "notation" for the ordinal $| a |_R$. The further structure we shall impose will be mainly designed to insure that successor and limit notations should be effectively identifiable as such. Thus we shall require that

(1) If $| a |_R = \alpha + 1$, then $a = 2^b$, where b is such that $| b |_R = \alpha$.

(2) If $| a |_R$ is a limit number then a is not a power of 2.

(3) For all $a \in F(R)$, $2^a \in F(R)$.

Further restrictions will be mentioned as they are used. Note that in virtue of (1)–(3), we can effectively tell whether $a \in F(R)$ is a successor or a limit notation. For a is a successor notation iff a is a power of 2, and a is a limit notation otherwise, provided $a \neq 1$. Also, if a is a successor notation, its log to the base 2 is a notation for the predecessor, and it is easily shown from (1)–(3) that if a is any notation, then 2^a is a notation for the successor.

Received by the editors June 2, 1964.

(1) Work reported herein was supported by Project MAC, and M.I.T. research program sponsored by the Advanced Research Projects Agency, Department of Defense, under Office of Naval Research Contract Number Nonr-4102(01). Reproduction in whole or part is permitted for any purpose of the United States Government.

If no further restrictions are imposed, we have essentially the class of *systems of notations* in the sense of [E]. It is easily proved that for every limit ordinal α in the classical second number class, a system $<_R$ exists with $|<_R| = \alpha$.

Following Enderton in a slight modification of the well-known Davis-Mostowski-Spector definition of the sets H_a, we define(2)

(1) $H^R(1) = \varnothing$.

(2) For $a \in F(R)$, $H^R(2^a) = (H^R(a))'$, where the accent denotes the ordinary jump [D, Definition 4.3, p. 75].

(3) For $a \in F(R)$, a not a power of 2, $a \neq 1$,

$$H^R(a) = \{J(b,c) \colon b \in H^R(c) \,\&\, c <_R a\}(^3).$$

We also write $H(a)$ for $H^R(a)$ when it is clear which system $<_R$ is under consideration. Familiarity with the notation of [D] and [K2] is presupposed. We shall write $S \leqq_T R$ to indicate ordinary (Turing) reducibility, in place of Davis' notation $S \prec R$.

2. **Introduction.** Enderton calls a system of notations $<_R$ *minimal* (with respect to the ordinary jump operation) if for every system $<_S$: $a \in F(R)$, $b \in F(S)$, $|a|_R \leqq |b|_S$ implies $H^R(a) \leqq_T H^S(b)$. Minimal systems have the highly desirable property that the Turing degrees of the H-sets are thus "as low as possible." Minimal systems, when they exist, enable us to associate degrees of unsolvability with ordinals(4) in such a way that

(1) d_0 (the degree associated with the ordinal 0) = the degree 0.

(2) $d_{\alpha+1} = (d_\alpha)'$ (for all α such that there exists a notation for α in a minimal system).

(3) If α is a limit ordinal for which a notation exists in a minimal system, then for all $\beta < \alpha$, d_β is defined and $d_\beta < d_\alpha$.

Moreover, if $<_R$ is an *arbitrary* system, and $|a|_R = \alpha$, then $H^R(a)$ has degree $\geqq d_\alpha$.

The main result of [E] is that $<_0$ is a minimal system. The present paper presents two results. First of all, we shall establish that *no* minimal system contains a notation for ω_1 (the least nonconstructive ordinal). Thus $<_0$ is a *maximal* minimal system—i.e., $<_0$ contains notations for as large a segment of the classical ordinals as is possible for a minimal system to do. This is a striking illustration of the great naturalness of the class of constructive ordinals. Secondly, we shall show that every D-system in the sense of [P1] is *almost-minimal*, in the sense that if $<_R$ is a D-system, $<_S$ is an arbitrary system, $a \in F(R)$, $b \in F(S)$, $|a|_R \leqq |b|_S$,

(2) Enderton has shown that in the case of the hyperarithmetic hierarchy defined over $<_0$ $H_a \equiv_T H(a)$ for all $a \in 0$ (see Theorem 2 [E]).

(3) In the sequel J is the usual "pairing" function (cf. [D], pp. 43–44).

(4) To do this, one simply defines d_α, the degree associated with α, to be the degree of unsolvability of $H^R(a)$ where $|a|_R = \alpha$ and $<_R$ is any minimal system which contains a notation for α.

then $H^R(a) \leqq_T H^S(2^b)$. Since all so far presented systems (e.g., the system C discussed in [W], [KR], [P1)], are D-systems or easily transformed into D-systems, the significance of these two results together can be put in the following way: There is no guarantee that systems extending $<_0$ will assign the same degrees of unsolvability to ordinals $\geqq \omega_1$. However, any system meeting the rather weak requirements for a D-system will assign *almost* the same degrees as any other, in the sense that if $<_R$, $<_S$ are both D-systems and $|a|_R = |b|_S$, then $H^R(a) \leqq_T H^S(2^b)$ and $H^S(b) \leqq_T H^R(2^a)$. In this sense, we have external almost-uniqueness(5) as far as there are D-systems.

3. Failure of minimality at ω_1.

THEOREM 1. *If K is any non-hyperarithmetic set, there is a system of notations $<_S$ containing a notation a_1 for ω_1, such that the degree of $H_S(a_1)$ is incomparable with the degree of K.*

Immediately we have the following corollaries.

THEOREM 2. *If $<_R$ is a minimal system of notations, then $|<_R| \leqq \omega_1$.*

Proof. Suppose on the contrary that $<_R$ contains a notation for ω_1, say $b_1 \in F(R)$ and $|b_1|_R = \omega_1$. Then, since $<_0$ is minimal [E], $H^0(a) \leqq_T H^R(b_1)$ for all $a \in 0$, and $H^R(b_1)$ is not hyperarithmetic. Applying Theorem 1, $H_R(b_1) \not\leqq_T H_S(a_1)$, the H-set associated with ω_1 in $<_S$, and this contradicts the minimality of $<_R$. ▮

THEOREM 3. *If $<_R$ is a minimal system of notations, then for all $a \in F(R)$, $H^R(a) \in \Sigma_1^1 \cap \Pi_1^1$.*

Proof. For any $a \in F(R)$ there is a $b \in 0$ such that $|a|_R \leqq |b|_0$ (Theorem 2); and the minimality of $<_R$ implies that $H^R(a) \leqq_T H^0(b)$. ▮

In proving Theorem 1, we shall use a subset (denoted by RSN) of those recursive well-orderings which are also systems of notations.

Let W be the set of Gödel numbers of recursive (linear) well-orderings defined in [S]: f belongs to W if and only if f is the Gödel number of a (general) recursive function of two variables such that the relation

$$\leqq_f =_{\mathrm{df}} \{\langle x,y\rangle : \{f\}(x,y) = 0\}$$

is a linear well-ordering of the field

$$F(f) = \{x : \{f\}(x,x) = 0\}.$$

We define $x <_f y \Leftrightarrow \mathrm{df}\ x \leqq_f y,\ x \neq y$.

(5) A *uniqueness theorem* says that if $|a|_R = |b|_S$, then $H^R(a) \equiv_T H^S(b)$, where $<_R$, $<_S$ are certain not-necessarily distinct systems. If $R = S$, we have an *internal uniqueness* theorem (e.g., [S], theorem for $<_0$); if $R \neq S$, we have an *external* uniqueness theorem. Internal and external *almost-uniqueness* are similarly defined with "$H^R(a) \leqq_T H^S(2^b)$ & $H^S(b) \leqq_T H^R(2^a)$" replacing "$H^R(a) \equiv_T H^S(b)$".

RSN is obtained from W as follows. Let p_e be the eth prime number. For each $e \in W$ let e' be the ordering defined by(6),

$$p_{e+2}^{x+1} <_{e'} p_{e+2}^{y+1} \Leftrightarrow \text{df } x <_e y.$$

Essentially, e' is an ordering of powers of a prime greater than 3; notice $e' \in W$ and $| <_{e'} | = | <_e |$. Now consider the class of orderings obtainable by laying end-to-end any finite number of different orderings e': that is, for each n-tuple $(e'_1, e'_2, \cdots, e'_n)$ such that if $i \neq j$ then $e'_i \neq e'_j$, this class contains the ordering e'' defined over $F(e'_1) \cup F(e'_2) \cup \cdots \cup F(e'_n)$ by,

$$p_{e_i+2}^{x+1} <_{e''} p_{e_j+2}^{y+1} \Leftrightarrow \text{df } i < j \vee (i = j \,\&\, p_{e_i+2}^{x+1} <_{e'_i} p_{e_i+2}^{y+1}).$$

Since if $i \neq j$, $F(e'_i) \cap F(e'_j) = \varnothing$, the orderings e'' are well-defined. If e'' is any member of this class, let e''' be the recursive system of notations,

$$1, \cdots, 3^x, 2^{3^x}, 2^{2^{3^x}}, \cdots, 3^y, 2^{3^y}, 2^{2^{3^y}}, \cdots, \quad \text{where } x <_{e''} y.$$

Then RSN is the set of (Gödel numbers of) all such recursive systems of notations.

From the foregoing description it can easily be verified that $\text{RSN} \subset W$ and that

$$e \in \text{RSN} \equiv (\exists u)((\forall i < 1h(u))(u)_i \in W \,\&\, A(e,u)),$$

where $A(e,u)$ is a predicate containing only number quantifiers. Since $W \in \Pi_1^1$, only universal *function* quantifiers appear in the predicate on the right, and the quantifier manipulation devices of [K2] way be used to express RSN in the one-universal-function quantifier form. Hence $\text{RSN} \in \Pi_1^1$.

We shall use the following simple properties of RSN.

(i) If $e \in \text{RSN}$, then $<_e$ is a system of notations.

(ii) If $e \in \text{RSN}$, then $| <_e |$ is a limit ordinal.

(iii) For each ordinal $\gamma < \omega_1$ there is an $e \in \text{RSN}$ such that $| <_e | \geqq \gamma$.

(iv) For each triple $\langle e, \gamma, E \rangle$ where $e \in \text{RSN}$, γ is a recursive ordinal, and E is a finite set of integers such that $E \cap F(e) = \varnothing$, there is an $f \in \text{RSN}$ with the properties that $| <_f | \geqq \gamma$, $E \cap F(f) = \varnothing$, and the ordering $<_e$ is an initial segment of the ordering $<_f$. In this case we say that $<_f$ is an *extension* of $<_e$ *excluding E.* (E.g. suppose $<_e$ is obtained from $(e'_1, e'_2, \cdots, e'_n)$. Let e'_{n+1} be an ordering of powers of a prime greater than any member of E, and $| <_{e'n+1} | \geqq \gamma$, and $e'_{n+1} \neq e'_i$ for $i \leqq n$. Then the system of notations $<_f$ obtained from $(e'_1, \cdots, e'_n, e'_{n+1})$ is an extension of $<_e$ excluding E.)

(v) If $e \in \text{RSN}$, then for all $a \in F(e)$, $H^e(a) \in \Sigma_1^1 \cap \Pi_1^1$. This is easily proved by induction over $<_e$ in a manner similar to [K2, Theorem 9].

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(6) e' is a Gödel number of a general recursive function such that $\{e'\}(p_{e+2}^{u+1}, p_{e+2}^{v+1}) = 0 \Leftrightarrow \{e\}(u,v) = 0$.

We define for each $e \in \mathrm{RSN}$ a "sum-set"

$$H(<_e) = \{J(a,b): b \in F(e) \,\&\, a \in H^e(b)\}.$$

Clearly $H(<_e)$ is also hyperarithmetic. Indeed, if $<_f$ is an extension of $<_e$ and $a \in F(f)$ is such that $\leqq_e = \{\langle x,y\rangle: x \leqq_f y \,\&\, y <_f a\}$, then $H(<_e) = H_f(a)$.

Terminology. Below "$T_n^A(k) = 0$" abbreviates the predicate "$(\exists y)(T_1^A(n,k,y) \,\&\, U(y) = 0)$"—i.e., "the computation of Turing machine T_n with Gödel number n, relative to set A with input k, halts with output 0" [D,p.58]. Thus "Turing machine T_n decides B relative to A" means that "$x \in B \Leftrightarrow T_n^A(x) = 0$ and $x \in \bar{B} \Leftrightarrow T_n^A(x) = 1$". "$T_n$ gives the wrong answer about membership of x in B, relative to A" means "$x \in B \,\&\, T_n^A(x) = 1$ or $x \in \bar{B} \,\&\, T_n^A(x) = 0$."

Proof of Theorem 1. Let K be a nonhyperarithmetic set. We construct below a system of notations, $<_S$, containing 17 as a notation for ω_1; the degree of $H^S(17)$ is incomparable with the degree of K.

Let $\gamma_1, \gamma_2, \gamma_3, \cdots$ be an ascending sequence of ordinals with limit ω_1. Our construction requires an infinite number of steps. At the completion of the nth step we will have obtained two items: a recursive system of notations of ordinal at least γ_n as an initial segment of $<_S$, *and* a finite set of numbers E_n to be *prohibited* from appearing in $<_S$. Furthermore we will have *spoiled* the first n Turing machines from ever providing a decision procedure for K relative to $H(17)$ or vice versa. Basically, each step consists in choosing an extension of the so far obtained initial segment of $<_S$ which *excludes* the numbers already prohibited, and reaches the next ordinal in the sequence $\{\gamma_n\}$. As will become clear, such extensions always exist (and are so chosen) so that the next Turing machine is spoiled, and it is for this reason that we may be forced to add a finite number of new integers to the already prohibited set.

Let $T_1, T_2, T_3, \cdots$ be an enumeration of Turing machines. The construction of $<_S$ proceeds as follows.

Step $n+1$. Let $<_{e_n}$ be the segment so far obtained, and E_n the finite set of integers prohibited by the nth stage (if $n = 0$, these are both empty). Note that $E_n \cap F(e_n) = \varnothing$.

Consider T_{n+1}: First, T_{n+1} cannot decide K relative to $H(<_{e_n})$ since K is not hyperarithmetic.

Case (i). There is a least number k such that T_{n+1} gives the wrong answer about membership of k in K relative to $H(<_{e_n})$. Let $J(a_1, b_1), \cdots, J(a_p, b_p)$ be the members of $H(<_{e_n})$, and $J(a_1', b_1'), \cdots, J(a_q', b_q')$ be the members of $\overline{H(<_{e_n})}$ about which T_{n+1} inquired during its "wrong" computation. For each $J(a_i', b_i'), 1 \leqq i \leqq q$, *either* $b_i \in F(e_n)$ but $a_i' \notin H^{e_n}(b_i')$ *or* $b_i' \notin F(e_n)$.

First, prohibit the set $\{b_i' : b_i' \notin F(e_n)\}$ from appearing in $<_S$. Let $E_n' = E_n \cup \{b_i' : b_i' \notin F(e_n)\}$.

Secondly, choose $<_f$ as the $(n+1)$th segment of $<_S$, if f is the least member of RSN such that (i) f extends e_n excluding E_n', and (ii) $|<_f| \geqq \gamma_{n+1}$.

Case (ii). Case (i) does not apply, but there is a least member f of RSN such that (i) f extends e_n excluding E_n, (ii) $|<_f| \geqq \gamma_{n+1}$, and (iii) there is a least k such that T_{n+1} gives the wrong answer about membership of k in K relative to $H(<_f)$. As before, if $J(a'_1, b'_1), \cdots, J(a'_q, b'_q)$ are the members of $\overline{H(<_f)}$ inquired about during the "wrong" computation of T_{n+1}, prohibit the set $\{b'_i : b'_i \notin F(f)\}$, and choose $<_f$ as the $(n+1)$th segment of $<_S$.

Let $E'_n = E_n \cup \{b'_i : \notin F(f)\}$.

In either of Cases (i) or (ii) we say that T_{n+1} is *spoiled strongly.*

Case (iii). Neither of Cases (i) or (ii) hold. There are two possibilities.

First, there is a k such that for every extension f of e_n excluding E_n, T_{n+1} does not give any answer about the membership of k in K relative to $H(<_f)$. Then we choose as the $(n+1)$th segment of $<_S$, the least f extending e excluding E_n such that $|<_f| \geqq \gamma_{n+1}$. Let $E'_n = E_n$.

In this case we say that T_{n+1} is *spoiled weakly.*

Secondly, for every k there is an extension $f_{\tau(k)}$ of e_n which excludes E_n and is such that T_{n+1} answers correctly the membership question for k in K (and of course there is no extension giving a wrong answer since Case (ii) does not apply). We argue that this possibility cannot arise, for then K would be expressible as follows:

$$k \in K \Leftrightarrow (\exists f)\,(f \in \text{RSN} \;\&\; \text{“}f \text{ extends } e_n \text{ excluding } E_n\text{”} \;\&$$

$$T_{n+1}^{H(<_f)}(k) = 0) \Leftrightarrow (\exists f)\,[f \in \text{RSN} \;\&\; \text{“}f \text{ extends } e_n \text{ excluding } E_n\text{”} \&$$

$$(\forall \beta)(\forall \alpha)((\text{“}\alpha \text{ assigns } H\text{-sets to } <_f\text{”} \&$$

$$\beta = \{J(a,b) : b \in F(f) \;\&\; \alpha(a,b) = 1\}) \Rightarrow$$

$$T_{n+1}^{\beta}\;(k) = 0)],$$

where "f extends e_n excluding E_n" $\equiv (\forall x)(\forall y)(x \leqq_{e_n} y \Rightarrow x \leqq_f y) \& (\forall z)(z \in E_n \Rightarrow z \notin F(f)))$, and "$\alpha$ assigns H-sets to $<_f$" $=_{\text{df}} (\forall x)\alpha(x,1) \neq 1 \,\&\, (\forall z)(\{f\}(z,z) = 0 \Rightarrow (\forall x)(\alpha(x, 2^z) = 1 \Leftrightarrow (\exists y) T_1^P(x,x,y))) \;\&\; (\forall z)(\{f\}(z,z) = 0 \;\&\; (\exists w > 0)(z = 3^w) \Rightarrow (\forall x)((\alpha(x,z) = 1 \Leftrightarrow (\exists u)(\exists v)(\{f\}(v,z) = 0 \;\&\; v \neq z \;\&\; \alpha(u,v) = 1 \;\&\; x = J(u,v))))$, where "$P$" appreviates "$\{u : \alpha(u,z) = 1\}$". Here, α is simply a mapping from $F(f)$ onto the H-sets defined over $<_f$. such that if $a \in F(f)$ then $H^f(a) = \{x : \alpha(x,a) = 1\}$.

Therefore $k \in K \Leftrightarrow (\exists f)\,(\Pi_1^1 \& \Pi_1^0 \& \Pi_1^1)$, and by bringing out the function quantifiers and contracting, we have $K \in \Pi_1^1$.

But $\bar{K}$ is similarly expressible in Π_1^1-form by replacing "$T_{n+1}^{H(<_f)}(k) = 0$" by "$T_{n+1}^{H(<_f)}(k) = 1$" in the predicate above; which is impossible since K is not hyperarithmetic.

Therefore, for every n, T_n can be spoiled either strongly or weakly.

Case (iv). Let f be the extension of e_n already chosen. If there is a pair (a,b) such that $b \notin F(f)$ and $T_{n+1}^K(J(a,b)) = 0$, prohibit the least such b; let $E_{n+1} = E'_n \cup \{b\}$.

Other let $E_{n+1} = E_n'$. (This is sufficient to spoil T_{n+1} from deciding $H(17)$ relative to K since for every initial segment, $<_f$, of $<_S$ there is a pair (a, b) such that $J(a,b) \in H(17)$ and $b \notin F(f)$.)

Let $<_R$ be the limit of the orderings $<_{e_n}$ in the obvious sense, i.e.,

$$x <_R y \equiv (\exists n)(x <_{e_n} y),$$

and define

$$<_S =_{\mathrm{df}} \{\langle x, y \rangle : x <_R y \ \vee \ (\exists z)(x <_R z \,\&\, y = 17)\}.$$

It is easily seen that $<_S$ is indeed a system of notations, and that $|17|_S = \omega_1$. Therefore we need only show that $K \not\leq_T H(17)$.

Let T_n be spoiled strongly. There is an initial segment $<_f$ and a k such that $T_n^{H(<_f)}(k)$ gives the wrong answer to the question of membership of k in K. But (see Cases (i) and (ii)) $J(a_i, b_i) \in H(17)$, $1 \leqq i \leqq p$, and $J(a_i', b_i') \in \overline{H(17)}$, $1 \leqq i \leqq q$. Hence the computation $T_n^{H(17)}(k)$ gives the same wrong answer.

Let T_n be spoiled weakly on k at segment $<_{e_n}$ (Case (iii)). If T_n decides K relative to $H(17)$, the computation $T_n^{H(17)}(k)$ must halt (i.e., give an answer). Let $J(a_1,b_1), \cdots, J(a_p, b_p)$ be the members of $H(17)$ inquired about. Now, the finite number of b_i must all occur in the field of some initial segment of $<_S$. Indeed, they must all occur in the field of some extension of $<_{e_n}$ which excludes E_n, say $<_f$. But, then $J(a_i, b_i) \in H(<_f)$, $1 \leqq i \leqq p$, and $\overline{H(17)} \subset \overline{H(<_f)}$, and therefore $T_n^{H(<_f)}(k)$ halts. This implies that T_n was not spoiled weakly on k at e_n, contradiction.

4. **Almost minimal systems.** All minimal systems of notations stop short of ω_1. What sort of chaos exists among hierarchies which assign Turing degrees to ordinals beyond ω_1? If the method of constructing a hierarchy of degrees by induction over a system of notations is to be at all useful beyond ω_1, one must hope that some extended systems (perhaps even some of those already in the literature) will exhibit a "quasi-minimality" (e.g. we might require of $<_R$ the property: For any system $<_S$, if $a \in F(R)$, $b \in F(S)$ and $|a|_R = |b|_S$, then $H^R(a)$ is arithmetic in $H_S(b)$)(7).

We provide a partial answer as follows: Any system of notations having a definition which conforms to a certain general scheme (essentially transfinite induction with arithmetic clauses) is *almost-minimal* (cf. §2).

For these systems (D-systems) the situation is as good as one could hope for namely, at any ordinal, the degree of the hierarchy over a D-system is at most *one* jump above minimal. We shall use the following notation.

The set $\{a : |a|_R = \alpha\}$ of all notations for α in $<_R$ will be denoted by N_α. Similarly R_α denotes the set of notations in $<_R$ for ordinals less than α, and $<_R^\alpha$ denotes the initial segment of $<_R$ restricted to R_α—i.e. $x <_R^\alpha y \Leftrightarrow x <_R y \,\&\, |y|_R < \alpha$.

(7) The system $<_0^0$ of [K3] is not minimal nor even quasi-minimal; it is however minimal for hierarchies of hyperdegrees [E].

A system of notations is regarded as "constructive" if it has, in some sense, a constructive definition. In the past, this has usually been an "inductive" definition of one of two kinds. Either an induction over the classical ordinals explicitly (as S_1 [K1] or C [P1]), or else an induction the clauses of which do not refer to the ordinals (and in actual fact are generally no more than conditions which admit more than one solution), accompanied by an "extremal" clause of an essentially impredicative nature (to determine which solution is meant, e.g. S_3 [K1], [K2], [K3]. A definition of the second kind is easily replaced by one of the first kind. Intuitively, a definition of $<_R$ of the first kind may be regarded as a step by step construction of the sets of notations N_α and the initial segments $<_R^\alpha$ simultaneously: If $<_R^\alpha$ has already been defined, the definition then "extends" the system, to contain notations for $\alpha + 1$, by defining $N_{\alpha+1}$ and $<_R^{\alpha+1}$ in terms of $<_R^\alpha$. We study systems whose methods of extension are arithmetic operations in the sense of [P2]:

DEFINITION 1. An arithmetic operation is any formula of second-order arithmetic containing one free variable P for a two-term relation between numbers, free variables for numbers, arbitrary bound number variables, but no free or bound higher-order variables except free P (the constants are to designate numbers and recursive functions and predicates).

DEFINITION 2. *A system of notations* $<_D$ is a *D-system* if there are arithmetic operations θ_D, ϕ_D, ψ_D such that the sequence $\{N_\alpha\}$ of sets of notations and the sequence $\{<_D^\alpha\}$ of initial segments, satisfy the following conditions (α, β are ordinals):

(1) $N_0 = \{1\}$.

(2) For any successor ordinal $\alpha + 1$, let $<_D^{\alpha+1}$ be defined by

$$x <_D^{\alpha+1} y \Leftrightarrow (\exists \beta)(\beta \leqq \alpha \ \&\ y \in N_\beta \ \&\ x <_D y).$$

Then (a) $N_{\alpha+1} = \{2^x : x \in N_\alpha\}$ and (b) $y \in N_{\alpha+1} \Rightarrow x <_D y \equiv \psi_D(x, y, <_D^{\alpha+1})$.

(3) $x \in F(D) \Rightarrow x <_D 2^{x}$

(4) $x <_D y \ \&\ y <_D z \Rightarrow x <_D z$.

(5) For any limit ordinal α, let $<_D^\alpha$ be defined by

$$x <_D^\alpha y \Leftrightarrow (\exists \beta)(\beta < \alpha \ \&\ y \in N_\beta \ \&\ x <_D y).$$

Then (a) $N_\alpha = \{x : \theta_D(x, <_D^\alpha)\}$ and (b) $y \in N_\alpha \Rightarrow x <_D y \equiv \phi_D(x, y, <_D^\alpha)$.

The fact that $<_D$ is required to be a system of notations immediately imposes certain restraints on the θ_D, ϕ_D, and ψ_D:

(a) The sets N_α must be disjoint,

(b) If α is a limit ordinal, N_α does not contain a power of 2.

(c) successor notations are of the form 2^x, and $|x|_D + 1 = |2^x|_D$.

(d) 1 is the unique minimal element.

An immediate consequence is that $|<_D|$ is a countable limit ordinal—i.e. the ordinal α such that $N_\beta \neq \varnothing$ for $\beta<\alpha$ and $N_\beta=\varnothing$ for $\alpha \leqq \beta$. Further, it is easily verified that conditions (1)–(5) above provide a simultaneous inductive definition of $<_D$ and the sets N_α.

The essential point of the definition is that at any ordinal α, the system can be extended using only *arithmetic* statements about the segment $<_D^\alpha$ already obtained. It is easy to see that the systems S_1 and S_3 of [K1], [K2], [K3] are D-systems. More to the point (since we are primarily interested in the cases where $|<_D| > \omega_1$), the systems O_{2_0} [Addison-Kleene] and $\tilde{C}$ [KR] extending S_3 are also D-systems. We have included clause (2b) so that S_1 and extensions of it such as C [W], [KR], [P1], [P2] will belong to the class of D-systems (in these cases the role of ψ_D will be to impose the natural ordering, $x <_D y \Leftrightarrow |x|_D < |y|_D$ at successor ordinals) (8).

Each D-system may be used to define an ascending sequence of degrees by employing the H^D-sets. Our purpose is to compare the degrees in any such sequence with the degrees of a sequence of H-sets defined over an arbitrary system of notations. This comparison is made easier by defining a second hierarchy of degrees over each D-system as follows (we abbreviate "a is a limit" by lim(a)):

(1) $\mathscr{H}^D(1)=\varnothing$.

(2) If $a \in F(D)$, $\mathscr{H}^D(2^a) = (\mathscr{H}^D(a))'$.

(3) If $a \in F(D)$ and lim(a), then

$$\mathscr{H}^D(a) = \{J(x,y): x \in \mathscr{H}^D(y) \;\&\; |y|_D < |a|_D\} (9).$$

We shall refer to the $\mathscr{H}^D$-sets as the "dense" $\mathscr{H}$-sets in opposition to the H^D-sets (sometimes called the "sparse" H-sets). The dense $\mathscr{H}$-sets have the following properties:

(i) If $|a|_D = |b|_D$ then $\mathscr{H}^D(a) = \mathscr{H}^D(b)$.

(ii) If $|a|_D < |b|_D$ and lim(b), then $\mathscr{H}^D(a)$ is recursive in $\mathscr{H}^D(b)$, uniformly in a (since $x \in \mathscr{H}^D(a) \Leftrightarrow J(x,a) \in \mathscr{H}^D(b)$).

Further, a simple relationship between $<_D$ and the hierarchy of dense $\mathscr{H}^D$-sets is provided by Lemma 1 below.

Notation. $\{e\}$ denotes the partial recursive function (p.r.f.) with Gödel number (g.n.) e. $A = \{e\}^B$ means "A is recursively enumerable in B with Gödel number e",—i.e. $x \in A$ if and only if the computation $\{e\}^B(x)$ halts. $A = [e]B$ means $A \leqq_T B$ with Gödel number e.

Also, for convenience we introduce the notation x^* for 2^x, x^{**} for 2^{2^x} and so on; that is, if $x \neq 0$, then $x^{*0} = x$ and $x^{*n+1} = 2^{x^{*n}}$.

Thus, $(\mathscr{H}(x))' = \mathscr{H}(x^*)$ and $(\mathscr{H}(x))^{(m)} = \mathscr{H}(x^{*m})$.

LEMMA 1. *For any D-system $<_D$ there is a p.r.f. $\{f\}$ such that if $a \in F(D)$ and* lim(a), *then $<_D^{|a|}$ is recursive in $\mathscr{H}^D(a)$ with Gödel number $\{f\}(a)$.*

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(8) The definition of D-system given here differs from the one given in [P2], in order to include systems in which the ordering $<_D$ is *not* the natural ordering.

(9) In the case of S_1-type systems, $\mathscr{H}(a) = H(a)$.

The proof of this lemma (and also of Theorem 4 below) uses transfinite induction in a form given by the recursion lemma [R], [E]:

RECURSION LEMMA. *Let $<_R$ be a partial well-ordering. Let P be a two-place predicate, and let K be a p.r.f. such that for any $a \in F(R)$ and any e:*

$$(\forall b <_R a)\ P(\{e\}(b), b) \Rightarrow P(K(e,a), a).$$

Then there is a p.r.f. $\{r\}$ (and r depends effectively on a Gödel number for K) with the property that for all $a \in F(R)$, $P(\{r\}(a), a)$.

Proof. There is an e_0 (effectively computable from a Gödel number for K) such that $\{e_0\} = \lambda a K(e_0, a)$ (recursion theorem ([IM], p. 352). If K has the property stated, then for all $a \in F(R)$, $P(\{e_0\}(a), a)$ follows by transfinite induction over $<_R$.

We introduce some special functions and Gödel numbers which will enable us to supply details in the proofs below:

For each D-system there must be integers l, m, n such that for all sets A, $\theta_D(x,A) \leqq_T A^{(l)}$, $\phi_D(x,y,A) \leqq_T A^{(m)}$ and $\psi_D(x,y,A) \leqq_T A^{(n)}$ since θ_D, ϕ_D, ψ_D are arithmetic. Taking l_0, m_0, n_0 to be the least such integers, let σ, ρ, τ be Gödel numbers such that $\theta_D(x,A) = [\sigma]A^{(l_0)}$, $\phi_D(x,y,A) = [\rho]A^{(m_0)}$ and $\psi_D(x,y,A) = [\tau]A^{(n_0)}$.

$\{e_1\}$ a primitive recursive function such that for all sets A,B,C,

if $A = [x]B^{(m)}$ and $B = [y]C$, then $A = [\{e_1\}(x,y)]C^{(m)}$.

Proof of Lemma 1. We apply the recursion lemma to $<_D$ using the predicate $P(e,b) = \text{df } \lim(b) \Rightarrow <_D^{|b|} = [e]\ \mathscr{H}(b)$.

The problem, therefore, is to construct a p.r.f. K so that whenever the induction *hypothesis*, $(\forall b <_D a)\ P(\{e\}(b), b)$ holds, we may conclude that $P(K(e,a), a)$ also holds.

Assume that $(\forall b <_D a)\ P(\{e\}(b), b)$ holds, and that a is a limit notation. We now describe a procedure T which will decide if $x <_D^{|a|} y$ using $\mathscr{H}(a)$ as an oracle.

First,

$$\begin{aligned} x <_D^{|a|} y &\Leftrightarrow x <_D y \,\&\, |y| < |a| \\ &\Leftrightarrow x <_D^{|y^*|} y \,\&\, |y| < |a|. \end{aligned}$$

Now if $|y| < |a|$, the relation $<_D^{|y^*|}$ may be reduced to $\mathscr{H}(a)$ uniformly in e and y by the following argument (assume for convenience that $l_0 \leqq m_0 \leqq n_0$):

Let $y = z^{*i}$ where $\lim(z)$ (the case $y = 1^{*i}$ is trivial). Then $<_D^{|z|} = [\{e\}(z)]\mathscr{H}(z)$ by hypothesis. Now $u <^{|z^*|} v \equiv u <_D^{|z|} v \vee (\theta_D(v, <_D^{|z|}) \,\&\, \phi_D(u, v, <_D^{|z|}))$ and since $\theta_D(v, <_D^{|z|}) = [\{e\}(\sigma, \{e\}(z))]\mathscr{H}(z^{*l_0})$, $\phi_D(u, v, <_D^{|z|}) = [\{e\}(\rho, \{e\}(z))]\mathscr{H}(z^{*m_0})$, we can find a g.n. of $<_D^{|z^*|}$ in $\mathscr{H}(z^{*m_0})$.

Suppose $<_D^{|z^*|} = [\{g\}(e,z)]\mathscr{H}(z^{*m_0})$.

$$u <_D^{|z^{**}|} v \equiv u <_D^{|z^*|} v \vee (v = w^* \& \theta_D(w, <_D^{|z|}) \& \psi_D(u, v, <_D^{|z^*|})).$$

Thus, using the previous procedure for $<_D^{|z^*|}$, and

$$\psi_D(u, v, <_D^{|z^*|}) = [\{e_1\}(\tau, \{g\}(e,z))]\,\mathscr{H}(z^{*(m_0+n_0)}),$$

we can find a g.n. of $<_D^{|z^{**}|}$ in $\mathscr{H}(z^{*(m_0+n_0)})$. Similarly, by repeating this last step for $<_D^{|z^{*3}|}$, and so on, we can find a g.n. for $<_D^{|y^*|}$ in $\mathscr{H}(z^{*(m_0+in_0)})$, uniformly in e and y. But if $|y| < |a|$, then $\mathscr{H}(z^{*(m_0+in_0)}) \leqq_T \mathscr{H}(a)$, and we can find (uniformly in e and y) a g.n. of $<_D^{|y^*|}$ in $\mathscr{H}(a)$.

Further, we may decide the question, $|y| < |a|$ using $\mathscr{H}(a)$ as an oracle as follows:

Let k_0 be a number belonging to the jump of every set, i.e., for all sets $A, k_0 \in A'$. Then

(i) $k_0 \in \mathscr{H}(2)$.

(ii) If $\lim(a)$, then $J(k_0,2) \in \mathscr{H}(a)$.

(iii) If $\lim(a)$, then

$$|y| < |a| \Leftrightarrow y = 1 \vee J(k_0, y) \in \mathscr{H}(a) \vee J(J(k_0,2), y) \in \mathscr{H}(a).$$

Let T be a procedure which decides first if $|y| < |a|$ using $\mathscr{H}(a)$; and if so, then settle the question $x <_D^{|a|} y$ by using the procedure indicated above which decides $x <_D^{|y^*|} y$ relative to $\mathscr{H}(a)$.

Finally, we define $K(e,a)$ to be a Gödel number of T. ∎

COROLLARY 1. *For any D-system there is an integer l_0 such that if $a \in F(D)$ and* $\lim(a)$, *then $N_{|a|}$ is recursive in $\mathscr{H}^D(a^{*l_0})$ uniformly in a.*

Proof. Let l_0 be the integer such that for all sets A, $\theta_D(x,A) = [\sigma]A^{(l_0)}$. Then, since $\lim(a)$,

$$x \in N_{|a|} \equiv \theta_D(x, <_D^{|a|}) \leqq_T \mathscr{H}^D(a^{*l_0}),$$

uniformly in a, by Lemma 1.

COROLLARY 2. *For any D-system, if $a \in F(D)$ then $H^D(a) \leqq_T \mathscr{H}^D(a)$ uniformly in a.*

Proof. We apply the recursion lemma with the predicate, $P(e,a) =_{\text{df}} H(a) = [e]\,\mathscr{H}(a)$.

Assuming the induction hypothesis, $(\forall b <_D a)\ P(\{e\}(b), b)$, we may construct a p.r.f. K so that $P(K(e,a),a)$ as follows.

Let e_2 be a uniform Gödel number of the empty set, $\varnothing = [e_2]A$ for all sets A.

Let j be a primitive recursive function such that for all sets A, B, if $A = [x]B$, then $A' = [j(x)]B'$.

(i) $a = 1$. Define $K(e,1) = e_2$.

(ii) $a = 2^b$. Then $H(b) = [\{e\}(b)]\ \mathscr{H}(b)$ by hypothesis; hence, define $K(e,a) = j(\{e\}(b))$.

(iii) $\lim(a)$. Now, $J(x,y) \in H(a) \Leftrightarrow y <_D a \,\&\, x \in H(y)$. In this case, a procedure may be constructed which decides $H(a)$ relative to $\mathscr{H}(a)$: Given $J(x,y)$, first apply the procedure of Lemma 1 to decide if $y <_D a$ (using $\mathscr{H}(a)$ as an oracle). If $y <_D a$, then the question $x \in H(y)$ may be decided relative to $\mathscr{H}(a)$ since $H(y) \leqq_T \mathscr{H}(y)$ with g.n. $\{e\}(y)$ (by hypothesis) and $\mathscr{H}(y) \leqq_T \mathscr{H}(a)$ uniformly in y. Let $K(e,a)$ be a g.n. of this procedure. ▌

THEOREM 4. *Let $<_R$ be a system of notations. For any D-system, $<_D$. there is a p.r.f. $\{f\}_R$ such that for all $b \in F(D)$ and $a \in F(R)$, $|b|_D = |a|_R \Rightarrow \mathscr{H}^D(b) = \{\{f\}(a)\}^{H^R(a)}$.*

Essentially, the degree of the dense $\mathscr{H}$-set hierarchy over a D-system is recursively enumerable in the degree (at the same ordinal) of the sparse H-set hierarchy over an arbitrary system of notations.

Proof. First let us introduce the following special functions and Gödel numbers:

$\{e_3\}$ — a primitive recursive function such that for all sets A,B, if $A = \{x\}^B$ then $A' = \{\{e_3\}(x)\}^{B'}$.

$\{e_4\}$ — a primitive recursive function such that for any system $<_R$ and any set A, if $a,b, \in F(R)$, $\lim(a)$, $b <_R a$ and $A = \{x\}^{H(b)}$, then $A = [\{e_4\}(x,b^*)]H(a)$. Such a function exists since $A \leqq_T H(b^*)$ uniformly in x, and $H(b^*) \leqq_T H(a)$ uniformly in b^*.

$\{e_5\}$ — a primitive recursive function such that for sets A,B,C, if $A = [x]B$ and $B = [y]C$, then $A = [\{e_5\}(x,y)]C$.

$\{e_6\}$ — For a given D-system, let e_6 be a g.n. such that if a is a limit notation in $F(D)$, then $N_{|a|} = [e_6]\ \mathscr{H}^D(a^{*l_0})$.

We apply the recursion lemma to $<_R$ with the predicate,

$$P(e,a) =_{\text{df}} (\forall b)(|b|_D = |a|_R \Rightarrow \mathscr{H}^D(b) = \{e\}^{H^R(a)})$$

(this is reasonable in view of the uniqueness property that if $|c| = |b|$ then $\mathscr{H}(c) = \mathscr{H}(b)$).

The induction hypothesis is $(\forall c <_R a)\, P(\{e\}(c), c)$. The construction of the p.r.f. K now proceeds by cases; in each case the hypothesis clearly implies $P(K(e,a),a)$. (For convenience we shall omit the superscript D from $\mathscr{H}^D$ and the superscript R from H^R.)

Case (i). $a = 1$. Since $\mathscr{H}(1) = H(1) = \varnothing$ we simply define $K(e,1) = e_2$.

Case (ii). $a = c^*$. Let $|d|_D = |c|_R$ and $b = d^*$. Then $\mathscr{H}(d) = \{\{e\}(c)\}^{H(c)}$

by hypothesis. Therefore, $\mathscr{H}(b) = \{\{e_3\}(\{e\}(c))\}^{H(a)}$. Hence, define $K(e,a) = \{e_3\}(\{e\}(c))$ if $a = c^*$.

Case (iii). lim(a). Let $|b|_D = |a|_R$. If $|a|_R$ is a limit of limit ordinals (e.g. ω^2), then $\mathscr{H}(b)$ is merely the union of $\mathscr{H}$-sets at limit notations less than b. However, if $|a|_R$ is not a limit of limit ordinals, the situation is slightly more complicated. In this case there is a limit notation d, such that $|d|_D + \omega = |b|_D$, and $\mathscr{H}(b)$ may be expressed as a union of previous $\mathscr{H}$-sets as follows:

$$\text{(U)} \qquad \mathscr{H}(b) = \mathscr{H}(d) \cup (\mathscr{H}(d) \times N_{|d|}) \cup \bigcup_{i>0} (\mathscr{H}(d^{*i}) \times N_{|d^{*i}|}),$$

where $\mathscr{H}(z) \times N_{|z|} = \{J(x,y)\colon x \in \mathscr{H}(z) \,\&\, y \in N_{|z|}\}$.

Hence, in order to enumerate $\mathscr{H}(b)$ using $H(a)$ as an oracle, it is sufficient to list the limit notations less than a (in $<_R$), and for each of these notations to begin listing the members of the union of sets in (U). Below we describe a procedure for doing this. At stages (iv) and (v) the method used to enumerate $N_{|d^{*i}|}$ relative to $H(a)$ (where d is now any limit notation such that $|d|_D < |a|_R$) depends on the simple observations that $N_{|d^{*i}|} = \{x\colon x = y^{*i} \,\&\, y \in N_{|d|}\}$ and that by Lemma 1 (Corollary 1) and the induction hypothesis, we have the following chain of reducibilities:

$$N_{|d|} \leqq_T \mathscr{H}(d^{*l_0}) \leqq_T H(c^{*l_0+1}) \leqq_T H(a), \qquad \text{where } |d|_D = |c|_R.$$

(i) Enumerate the set $\{x\colon x <_R a\}$ using $H(a)$. (This may be done, for example, by listing x if and only if $x = 1 \vee J(k_0, x) \in H(a) \vee J(k_0, 2), x) \in H(a)$, where for all sets $A, k_0 \in A'$.)

(ii) Suppose a new member occurs on the mth step of (i). Then, *for all so far obtained limit elements c and the least element*, 1, carry out the following enumeration procedures (for any member c of this set, let d be such that $|d|_D = |c|_R$; this is merely a notational convenience—we do not need to compute a value for d).

(iii) List the first m members of each $\mathscr{H}(d)$ using $\mathscr{H}(d) = [\{e_4\}(\{e\}(c), c^*)]\, H(a)$.

(iv) List the first m members of each set $\mathscr{H}(d) \times N_{|d|}$ using some uniform procedure which forms $J(x,y)$ whenever x is listed by the procedure for computing $\mathscr{H}(d)$ given in (iii), and y is listed by the procedure for computing $N_{|d|}$ given by

$$N_{|d|} = [\{e_5\}(e_6, \{e_4\}(\{e\}(c^{*l_0}), c^{*l_0+1}))]\ H(a).$$

(v) List the first m members of $\mathscr{H}(d^{*i}) \times N_{|d^{*i}|}$ for each $i \leqq m$ (and all so far obtained limit elements c) by forming $J(x, y^{*i})$ whenever y is listed as a member of $N_{|d|}$ (by the procedure of (iv)) and x is listed as a member of $\mathscr{H}(d^{*i})$ by the procedure, $\mathscr{H}(d^{*i}) = [\{e_4\}(\{e\}(c^{*i}), c^{*i+1})]\ H(a)$. Also list the first m members $\mathscr{H}(1^{*i}) \times \{1^{*i}\}$ for each $i \leqq m$.

Finally, continue with the enumeration of step (i).

Clearly, if the induction hypothesis holds for $\{e\}$, every member of $\mathscr{H}(b)$, $|b|_D = |a|_R$, will be listed sooner or later by the above procedure. Moreover, a

Gödel number for this procedure depends uniformly on the e_6 and l_0 of $<_D$ and e. Hence, let $\{p\}$ be a primitive recursive function such that $\{p\}$ (e, e_6, l_0) is a g.n. for the procedure; then the induction hypothesis implies

$$\mathscr{H}(b) = \{\{p\}(e, e_6, l_0)\}^{H(a)}.$$

Define, $K(e,a) = \{p\}$ (e, e_6, l_0) if $\lim(a)$.

This completes the definition of K by cases, and the theorem now follows from the recursion lemma applied to $<_R$. ▌

Corollary 1. *If $<_R$ is a system of notations and $<_D$ is a D-system, then for all $a \in F(D)$ and $b \in F(R)$, $|a|_D = |b|_R \Rightarrow H^D(a) \leqq_T H^R(b^*)$, uniformly in a, b.*

Proof. Theorem 4 and Lemma 1 (Corollary 2). ▌

This was refered to in §2 as the "almost-minimal" property of D-systems. As we have seen in §3, it is not possible to improve this result to "minimal" in the sense of Enderton. But, at least we can take advantage of this seemingly negative result to evaluate the various methods of defining hierarchies over extended systems of notations.

Corollary 2. *For any D-system, $<_D$, if $a \in F(D)$ then*

$$H^D(a) \leqq_T \mathscr{H}^D(a) \leqq_T H^D(a^*),$$

uniformly in a.

Indeed a stronger result is true. Assume as in [P2] that every D-system has associated with it two recursive functions g, k such that if $a \in F(D)$ and $\lim(a)$, then $g(a) \in F(D)$ and $|g(a)| \leqq |a|$, and $\{k(a)\}$ provides an order-preserving cofinal mapping from $D_{|g(a)|}$ into $D_{|a|}$. Then a natural generalization of the hyperarithmetic hierarchy may be obtained by, (i) $H_1^D = \varnothing$, (ii) $H_{2^a}^D = (H_a^D)'$, and (iii) if $a \in F(D)$ and $\lim(a)$, then

$$H_a^D = \{J(x,y) \colon y \in D_{|g(a)|} \;\&\; x \in H^D_{(k(a))(y)}\}.$$

(Notice that the dense $\mathscr{H}$-sets arise as the special case when g and $\{k(a)\}$ are the identity function). It can be proved that for all $a \in F(D)$, $H_a \leqq_T \mathscr{H}(a)$ and $\mathscr{H}(a)$ is recursively enumerable in H_a, uniformly in a (details in [L]).

In view of the closeness of the degrees of the sparse H-set and dense $\mathscr{H}$-set hierarchies, and the fact that the dense $\mathscr{H}$-set structure is easier to work with, there would seem to be little point in adhering to the sparse H-sets when it comes to defining hierarchies over almost-minimal (but not minimal) systems of notations(10).

(10) A proof is given in [L] that D-system ordinals are almost-unique—i.e. $|a|_D = |b|_D$ implies $H_a^D \leqq_T H_{b^*}^D$ uniformly in a, b. Whether D-system ordinals are uniqueness ordinals in the sense of [S] is an open question; as is shown in [P1], internal uniqueness does not necessarily break down at ω_1.

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